Announcements

1) HW #4 Due Thursday

The Inverse Laplace Transform

Given a (continuous) function g, we define the Inverse Laplace Transform $\mathcal{L}^{-1}(g)$ to be any function f with

$$\boxed{\mathcal{L}(f) = g}$$

Observation: (uniqueness)

There is at most one continuous function f with $\mathcal{L}(f) = g$ – we'll usually take such an f to be "the" inverse Laplace transform of g, it it exists. If no continuous f exists, we want one with minimal discontinuities.

Linearity

Since the Laplace transform is linear, so is the inverse Laplace transform:

$$\boxed{\begin{aligned} \mathcal{L}^{-1}(g+h) &= \mathcal{L}^{-1}(g) + \mathcal{L}^{-1}(h) \\ \mathcal{L}^{-1}(cg) &= c\,\mathcal{L}^{-1}(g) \end{aligned}}$$

for functions g, h and constant c.

Translates of the Heaviside Function

Recall the Heaviside function

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

$$\mathcal{L}(u)(s) = \frac{1}{s}, \quad s > 0.$$

If $a > 0$,

$$u(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a. \end{cases}$$

Let $u_a(t) = u(t-a)$. Then

$$\mathcal{L}(u_a)(s) = \int_0^\infty u_a(t) e^{-st}\, dt$$

$$= \int_0^\infty u(t-a) e^{-st}\, dt$$

$$= \int_a^\infty e^{-st}\, dt$$

$$\mathcal{L}(u_a)(s) = \lim_{x\to\infty} \left.\frac{e^{-st}}{-s}\right|_a^x$$

$(s>0)$

$$= \lim_{x\to\infty}\left(\frac{e^{-sx}}{-s} + \frac{e^{-sa}}{s}\right)$$

$$= \boxed{\frac{e^{-sa}}{s}}$$

One More Laplace Transform Property

If $\mathcal{L}(f)(s)$ exists for $s > a$,

$$\mathcal{L}(f(t-a)u(t-a))(s)$$
$$= e^{-as}\mathcal{L}(f)(s)$$

To show this:

$$\mathcal{L}(f_a \cdot u_a)(s)$$
$$= \int_0^\infty f_a(t)\,u_a(t)\,e^{-st}\,dt$$

$$\mathcal{L}(f_a u_a)(s) = \int_0^\infty f(t-a)\,u(t-a)\,e^{-st}\,dt$$

$$= \int_a^\infty f(t-a)\,e^{-st}\,dt$$

Let $y = t - a$. Then $\dfrac{dy}{dt} = 1$,

so substituting,

$$\int_0^\infty f(y)\,e^{-s(y+a)}\,dy$$

constant w.r.t. y

$$= \int_0^\infty f(y)\,e^{-sy}\,e^{-sa}\,dy$$

So

$$\mathcal{L}(f_a u_a)(s) = e^{-sa}\int_0^\infty f(y)e^{-sy}\,dy$$

$$= e^{-sa}\mathcal{L}(f)(s)$$

Solving the Brine Problem

$$\mathcal{L}(x)(s) = 600\left(\frac{1}{3s} - \frac{500}{3(500s+3)}\right) +$$

$$600\left(\frac{e^{-10s}}{3s} - \frac{500e^{-10s}}{3(500s+3)}\right) + \frac{15000}{500s+3}$$

Solve for X by using the inverse Laplace transform.

$$\mathcal{L}(x)(s) = \frac{200}{3}\cdot\frac{1}{s} - \frac{85{,}000}{500s+3}$$

$$+ 200\left(\frac{e^{-10s}}{s}\right) - 100{,}000\,\frac{e^{-10s}}{500s+3}$$

$$x(t) = \mathcal{L}^{-1}\left(\mathcal{L}(x)(s)\right)$$

$$= \mathcal{L}^{-1}\left(\frac{200}{3}\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{85000}{500s+3}\right)$$

$$+ \mathcal{L}^{-1}\left(200\frac{e^{-10s}}{s}\right) - \mathcal{L}^{-1}\left(100{,}000\frac{e^{-10s}}{500s+3}\right)$$

$$= \frac{200}{3}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - 170\,\mathcal{L}^{-1}\left(\frac{1}{s+\frac{3}{500}}\right)$$

$$+ 200\,\mathcal{L}^{-1}\left(\frac{e^{-10s}}{s}\right) - 200\,\mathcal{L}^{-1}\left(\frac{e^{-10s}}{s+\frac{3}{500}}\right)$$

using linearity of $\mathcal{L}^{-1}$

$$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1,$$

$$\mathcal{L}^{-1}\left(\frac{1}{s+\frac{3}{500}}\right) = e^{-\frac{3t}{500}}$$

$$\mathcal{L}^{-1}\left(\frac{e^{-10s}}{s}\right) = u_{10}(t)$$
$$= u(t-10)$$

$$\mathcal{L}^{-1}\left(\frac{e^{-10s}}{s+\frac{3}{500}}\right) = \mathcal{L}^{-1}\left(e^{-10s}\mathcal{L}\left(e^{-\frac{3t}{500}}\right)(s)\right)$$
$$= \underbrace{u_{10}(t)\, e^{-\frac{3(t-10)}{500}}}_{\mathcal{L}(f_a u_a)(s) = e^{-as}\mathcal{L}(f)(s)}$$

So

$$\boxed{x(t) = \frac{200}{3} - 170e^{-\frac{3t}{500}} + 200u(t-10) - 200u(t-10)e^{-\frac{3(t-10)}{500}}}$$

Continuous for $t \geq 10$.

Rectangular Window Function

Catalog of Laplace Transforms

1) $\mathcal{L}(1)(s) = \dfrac{1}{s}, \quad s>0$

2) $\mathcal{L}(t)(s) = \dfrac{1}{s^2}, \quad s>0$

3) $\mathcal{L}(t^n)(s) = \dfrac{n!}{s^{n+1}}, \quad s>0$

4) $\mathcal{L}(\sin(at)) = \dfrac{a}{s^2+a^2}, \quad s>0$

5) $\mathcal{L}(\cos(at)) = \dfrac{s}{s^2+a^2}, \quad s>0$

6) $\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$, $s > a$

7) $\mathcal{L}(u(t-a)) = \dfrac{e^{-sa}}{s}$, $s > 0$

$(a \geq 0)$

Properties of Laplace transform, cont.

3) (higher derivatives)

$$\mathcal{L}(f'')(s) = \int_0^\infty f''(t)e^{-st}\,dt$$

$$= \lim_{x\to\infty} \int_0^x f''(t)e^{-st}\,dt$$

Integrate by parts:

$u = e^{-st}$ $\qquad$ $v = f'(t)$

$du = -se^{-st}$ $\qquad$ $dv = f''(t)\,dt$

$$\int_0^x f''(t)e^{-st}\,dt$$

$$= e^{-st}f'(t)\Big|_0^x + s\int_0^x f'(t)e^{-st}\,dt$$

$$= e^{-sx}f'(x) - f'(0) + s\int_0^x f'(t)e^{-st}\,dt$$

If f' is of exponential order,

taking the limit as $x \to \infty$,

$$-f'(0) + s\,\mathcal{L}(f')(s)$$

If f is of exponential order,

$$\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0),$$

so

$$\mathcal{L}(f'')(s) = -f'(0) + s\big(s\mathcal{L}(f)(s) - f(0)\big)$$

$$= \boxed{s^2\mathcal{L}(f)(s) - sf(0) - f'(0)}$$